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A Generalization of Burr Type XII Distribution with Properties, Copula and Modeling Symmetric and Skewed Real Data Sets

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Abstract
The Burr Type XII distribution is commonly used in reliability analysis, survival analysis, and actuarial science to model the time-to-failure of a system or component. It is also used in finance to model the distribution of portfolio returns and in hydrology to model the frequency of extreme events. In this work, a new generalization of Burr type XII model is introduced and studied. The genesis of the new model is based on the family of Cordeiro et al. (2016). The new model generalizes at least eight important sub-models. The new density can be unimodal, symmetric and left skewed. Some useful properties related to the new model are derived. The Clayton Copula-based construction is used to generate many bivariate and multivariate type distributions. Graphically, we performed the simulation experiments to assess of the finite sample behavior of the estimations.

Keywords: Burr XII Distribution; Symmetric Density; Copula; Kaplan-Meier; Maximum Likelihood; Simulation; Modeling; Unimodal.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

1.Introduction

The Burr Type XII distribution is a probability distribution used in statistics to model continuous random variables. It is also known as the Singh-Maddala distribution or the generalized log-logistic distribution. The Burr Type XII distribution is commonly used in reliability analysis, survival analysis, and actuarial science to model the time-to-failure of a system or component. It is also used in finance to model the distribution of portfolio returns and in hydrology to model the frequency of extreme events. Burr (1942) introduced a new system of probability density function (PDF) (see Elderton, (1953) and Elderton and Johnson (1969)). This system is obtained by considering cumulative distribution functions (CDFs) which satisfying a certain differential equation. A special attention has been paid to one of these new models, namely, the Burr type XII distribution (for more details see Burr (1942), Burr (1968), Burr (1973), and Tadikamalla (1980)). The survival function (SF) of the two-parameter BXII distribution is given by

$$S_{a,b}(w) = 1 - H_{a,b}(w) = (w^{a}+1)^{-b}|_{w\geq 0}, \tag{1}$$

where $H_{a,b}(w)$ refer to CDF of the BXII distribution. The PDF corresponding to (1) can be written as

$$h_{a,b}(w) = ab\,w^{a-1}(w^{a}+1)^{-b-1}|_{w\geq 0}, \tag{2}$$

where both a and b are shape parameters. For $a = 1$, the BXII model reduces to the Lomax (Lox) or Pareto type II (PaII) model. For $b = 1$, the BXII model reduces to the log-logistic (LL) model. When $b \to \infty$ the BXII model reduces the two-parameter Weibull model, for $b \to \infty$ the BXII model reduces the one-parameter Weibull model.

For more details about the BXII model and its relations with other related models, namely: PaII (or Lox), LL, compound Weibull, gamma (Ga) and Weibull exponential (WE) distributions.

The Burr Type XII distribution is a probability distribution that is often used to model various phenomena in many fields, including finance, insurance, engineering, and reliability. This distribution is also known as the Singh-Maddala distribution, and it is a flexible distribution that can model a wide range of shapes, including symmetric and skewed distributions. The Burr Type XII distribution can be used to model the distribution of insurance claims, which can help insurers estimate the amount of money they need to set aside to cover potential losses. It is also used to model the distribution of asset returns in finance. The Burr Type XII distribution is often used to model the lifetime of mechanical and electronic components in engineering and reliability studies. It can help engineers estimate the failure rate and reliability of these components, which can be useful in designing and maintaining equipment. The Burr Type XII distribution can be used to model the distribution of response times to marketing campaigns, such as the time it takes for customers to make a purchase after receiving a promotional offer. The Burr Type XII distribution can be used to model the distribution of extreme values in environmental studies, such as the maximum rainfall or the maximum wind speed in a particular area. Overall, the Burr Type XII distribution is a versatile distribution that can be used in many different fields to model a wide range of phenomena.

In order to properly model and evaluate real-world data that cannot be fully represented by existing distributions, a new probability distribution may be required. New distributions are created for a variety of reasons, including as addressing certain data traits or characteristics, enhancing the precision of simulations or forecasts, or offering more adaptable modelling alternatives. New distributions can also result in improvements in statistical theory and its use. When current distributions fall short of accurately describing the properties of real-world data, a new probability distribution may be required. This may occur when the data exhibits distinct traits or properties that cannot be explained by pre-existing distributions, such as asymmetry, heavy tails, or multi-modality. To provide a better match to the data and boost the precision of statistical analysis, forecasts, or simulations under these circumstances, a new distribution may be created. Depending on the particular application or issue being solved, a new distribution's motive can change. To represent the distribution of financial returns or exceptional occurrences, for instance, new distributions may be created in the field of finance. To model the distribution of gene expression levels in biology, new distributions may be created. New distributions may be created in engineering to simulate the distribution of material strength or fatigue life. The goal of creating a new distribution is ultimately to offer a more precise and adaptable tool for modelling and analyzing data, which can result in a better comprehension of the underlying mechanisms and improved decision-making. To determine whether a new distribution can accurately characterize the data and forecast the future, it is crucial to thoroughly assess its attributes and compare them to those of existing distributions. Depending on the individual situation and the type of data being examined, we may or may not require a new probability distribution. Existing distributions could be sufficient for modelling and analysis in some circumstances. To develop better forecasts or simulations or to offer more versatile modelling possibilities, it occasionally becomes necessary to create a new distribution in order to properly describe the data's properties. Consequently, it is important to assess each situation's need for a new probability distribution. Cordeiro et al. (2016) investigated a new flexible class of continuous distributions called the generalized odd log-logistic-G (GOLL-G) family with only two extra shape parameters. In the work, we introduce a new version of the BXII model using the GOLL-G family called the generalized odd log-logistic BXII (GOLLBXII). For an arbitrary baseline CDF $H_{\underline{\xi}}(w)$, the CDF of the GOLL-G family is given by

$$F_{\underline{\xi}}(w) = \frac{H_{\underline{\xi}}(w)^{\alpha\beta}}{H_{\underline{\xi}}(w)^{\alpha\beta} + \left[1 - H_{\underline{\xi}}(w)^{\beta}\right]^{\alpha}}, \tag{3}$$

where $\underline{\xi}$ is the parameter vector of the base line model and $H_{\underline{\xi}}(w)$ represents the CDF of the base line model. For $\beta = 1$ we get the OLL-G family (Gleaton and Lynch (2006)). For $\alpha = 1$ we get the proportional reversed hazard rate G (PRHR-G) family (Gupta and Gupta (2007)). The CDF of the GOLLBXII is given by

$$F_{\underline{\Psi}}(w) = \frac{\left[1-(w^{a}+1)^{-b}\right]^{\alpha\beta}}{\left[1-(w^{a}+1)^{-b}\right]^{\alpha\beta} + \left\{1-\left[1-(w^{a}+1)^{-b}\right]^{\beta}\right\}^{\alpha}}, \tag{4}$$

where $\underline{\Psi} = \alpha, \beta, \boldsymbol{a}, \boldsymbol{b}$. The PDF corresponding to (4) is given by

$$f_{\underline{\Psi}}(w) = \alpha\beta\boldsymbol{a}\boldsymbol{b}w^{\boldsymbol{a}-1}(w^{\boldsymbol{a}}+1)^{-\boldsymbol{b}-1}\left[1-(w^{\boldsymbol{a}}+1)^{-\boldsymbol{b}}\right]^{\alpha\beta-1}\left\{1-\left[1-(w^{\boldsymbol{a}}+1)^{-\boldsymbol{b}}\right]^{\beta}\right\}^{\alpha-1}$$
$$\times\left(\left[1-(w^{\boldsymbol{a}}+1)^{-\boldsymbol{b}}\right]^{\alpha\beta}+\left\{1-\left[1-(w^{\boldsymbol{a}}+1)^{-\boldsymbol{b}}\right]^{\beta}\right\}^{\alpha}\right)^{-2}. \quad (5)$$

The hazard rate function (HRF) for the GOLLBXII model can be obtained from $h_{\underline{\Psi}}(w) = f_{\underline{\Psi}}(w)/\left[1 - F_{\underline{\Psi}}(w)\right]$.
The creation of novel probability distributions is driven by a number of factors, including:

- **I.** Existing distributions may occasionally fall short of accurately describing the features of real-world data. Creating new distributions can aid in improving the data's representation and insight.
- **II.** To discuss particular aspects or properties of data.
- **III.** To address particular aspects of the data, such as skewness, heavy tails, or multimodality, new distributions may be created.
- **IV.** To increase simulation or prediction accuracy: In some circumstances, new distributions can offer more precision for projecting outcomes or simulating scenarios.
- **V.** To offer more adaptable modelling choices: The complexity of the data being investigated can be better captured by new distributions' more flexible modelling possibilities.
- **VI.** To advance statistical theory and applications: The development of new distributions can lead to advancements in statistical theory and have practical applications in various fields.

Many branches of mathematics, such as probability theory and statistics, depend on asymptotic features. The behavior of a mathematical function or a series of numbers when the input grows arbitrarily large or tiny is generally described by asymptotic characteristics. The asymptotic characteristics of statistical estimators and test statistics are of relevance in probability theory and statistics. Because they enable us to use huge sample sizes to draw conclusions from statistics, asymptotic characteristics are crucial. When the sample size grows, asymptotic results offer useful information about how a statistical estimator or test statistic behaves, and this knowledge can be utilized to derive significant statistical properties and draw conclusions about the population. Let $\varepsilon = inf\left\{w|_{[H_{\boldsymbol{a},\boldsymbol{b}}(w)>0]}\right\}$, the asymptotics of the CDF, PDF and HRF as $w \to \varepsilon$ are given by

$$F_{\underline{\Psi}}(w)|_{w\to\varepsilon} \sim \left[1-(w^{\boldsymbol{a}}+1)^{-\boldsymbol{b}}\right]^{\alpha\beta},$$

$$f_{\underline{\Psi}}(w)|_{w\to\varepsilon} \sim \alpha\beta\boldsymbol{a}\boldsymbol{b}w^{\boldsymbol{a}-1}(w^{\boldsymbol{a}}+1)^{-\boldsymbol{b}-1}\left[1-(w^{\boldsymbol{a}}+1)^{-\boldsymbol{b}}\right]^{\alpha\beta-1}$$

and

$$h_{\underline{\Psi}}(w)|_{w\to\varepsilon} \sim \alpha\beta\boldsymbol{a}\boldsymbol{b}w^{\boldsymbol{a}-1}(w^{\boldsymbol{a}}+1)^{-\boldsymbol{b}-1}\left\{1-(w^{\boldsymbol{a}}+1)^{-\boldsymbol{b}}\right\}^{\alpha\beta-1}.$$

The asymptotics of CDF, PDF and HRF as $w \to \infty$ are given by

$$1 - F_{\underline{\Psi}}(w)|_{w\to\infty} \sim \beta^{\alpha}(w^{\boldsymbol{a}}+1)^{-\alpha\boldsymbol{b}},\; f_{\underline{\Psi}}(w)|_{w\to\infty} \sim \alpha\beta^{\alpha}\boldsymbol{a}\boldsymbol{b}w^{\boldsymbol{a}-1}(w^{\boldsymbol{a}}+1)^{-\alpha\boldsymbol{b}-1}$$

and

$$h_{\underline{\Psi}}(w)|_{w\to\infty} \sim \alpha\boldsymbol{a}\boldsymbol{b}w^{\boldsymbol{a}-1}(w^{\boldsymbol{a}}+1)^{-1}.$$

Asymptotic features are significant in statistics because they enable the use of large sample sizes for statistical inference. They offer insightful data on how statistical estimators and test statistics behave as sample sizes grow, and this data can be utilized to deduce significant statistical properties and draw conclusions about the population. As the sample size increases significantly, the distribution of statistical estimators and test statistics is approximated using asymptotic theory. This makes it possible to approximate the statistic's exact distribution, which would otherwise be challenging to determine. Based on generalized binomial expansions and after some algebraic processes, the PDF in (6) can be rewritten as

$$f_{\underline{\Psi}}(w) = \sum_{\hbar=0}^{\infty} \quad C_{[\hbar]} \hbar_{[\boldsymbol{a},\boldsymbol{b}(1+\hbar)]}(w), \tag{6}$$

where

$$C_{[\hbar]} = \alpha\beta \sum_{d,j,q=0}^{\infty} \sum_{\kappa=0}^{q} (-1)^{j+\kappa+l+\hbar} \binom{-2}{d} \binom{-\alpha(d+1)}{j} \binom{\alpha\beta(d+1)+\beta j-1}{q} \binom{q}{\kappa} \binom{1+\kappa}{\hbar},$$

and $\hbar_{[\boldsymbol{a},\boldsymbol{b}(1+\hbar)]}(w)$ is the PDF of the BXII model with parameters $\boldsymbol{a}, \boldsymbol{b}(1+\hbar)$ where

$$\hbar_{[\boldsymbol{a},\boldsymbol{b}(1+\hbar)]}(w) = \boldsymbol{a}\boldsymbol{b}(1+\hbar) w^{\boldsymbol{a}-1} (w^{\boldsymbol{a}}+1)^{-\boldsymbol{b}(1+\hbar)-1}$$

So, the PDF of the GOLLBXII model can be expressed as a linear mixture of the BXII PDF. Via integrating (6), the CDF of the GOLLBXII model is

$$F_{\underline{\Psi}}(w) = \sum_{\hbar=0}^{\infty} \quad C_{[\hbar]} H_{[\boldsymbol{a},\boldsymbol{b}(1+\hbar)]}(w), \tag{7}$$

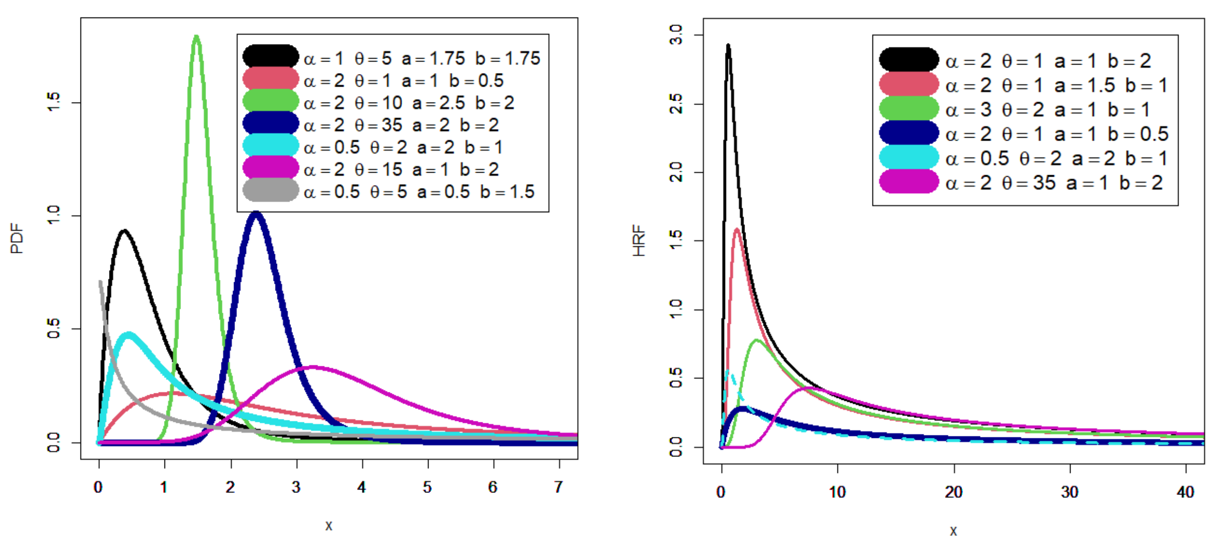
where $H_{[\boldsymbol{a},\boldsymbol{b}(1+\hbar)]}(w) = (w^{\boldsymbol{a}}+1)^{-\boldsymbol{b}(1+\hbar)}$ is the CDF of the BXII model with parameters $\boldsymbol{a}$, $\boldsymbol{b}(1+\hbar)$. Let W be a random variable (rv) having the BXII distribution (2) with parameters $\boldsymbol{a}_1, \boldsymbol{a}_2$. For $n < \boldsymbol{a}_1\boldsymbol{a}_2$, the r^{th} ordinary and incomplete moments of W are, respectively, given by $\mu_n' = \boldsymbol{a}_2 B\left(\boldsymbol{a}_2 - \frac{n}{\boldsymbol{a}_1}, \frac{\boldsymbol{a}_1+n}{\boldsymbol{a}_1}\right)$ and $I_n(t) = \boldsymbol{a}_2 B\left(t^{\boldsymbol{a}_1}; \boldsymbol{a}_2 - \frac{n}{\boldsymbol{a}_1}, \frac{\boldsymbol{a}_1+n}{\boldsymbol{a}_1}\right)$, where

$$B(\boldsymbol{a}_1, \boldsymbol{a}_2) = \int_0^{\infty} w^{\boldsymbol{a}_1-1} (1+w)^{-(\boldsymbol{a}_1+\boldsymbol{a}_2)} dw$$

and

$$B(z; \boldsymbol{a}_1, \boldsymbol{a}_2) = \int_0^{z} w^{\boldsymbol{a}_1-1} (1+w)^{-(\boldsymbol{a}_1+\boldsymbol{a}_2)} dw$$

are the beta and the incomplete beta functions of the second type, respectively. Table 1 provides some sub-models of the GOLLBXII model. Figure 1 give some plots for the new PDF and its corresponding HRF. It is noted from Figure 1(a) that the new PDF can be unimodal, symmetric and left skewed. From Figure 1(b) we note the that new HRF can be upside down shaped.



(a) (b)

Figure 1: Some plots for the new PDF and its corresponding HRF.

Due to Figure 1, the new can be considered as a heavy-tail model, probability distributions with "heavier" tails than the normal distribution are described to as heavy-tailed probability-based distributions; this suggests that they have a larger likelihood of discovering extreme values (outliers). In contrast, there is a minimal likelihood of detecting extreme values in light-tailed distributions, such the normal distribution. heavy-tailed distributions are important in many fields, including finance, economics, physics, and engineering, as they provide a more accurate representation of real-world phenomena where extreme events are more common than expected under a normal distribution.

Table 1: Sub-models of the GOLLBXII model.

N.	α	β	a	b	Reduced model
1	α	β	1	b	GOLLLx
2	α	β	a	1	GOLL-LL
3	α	1	a	b	OLL-BrXII
4	α	1	1	b	OLL-Lx
5	α	1	a	1	OLL-LL
6	1	β	a	b	PRHR-BXII
7	1	β	1	b	PRHR-Lx
8	1	β	a	1	PRHR-LL

2.Properties

Understanding probability distributions requires an understanding of mathematical properties. Mathematical functions called probability distributions describe the likelihood of various outcomes of a random variable. They are employed in a variety of fields, including as finance, physics, engineering, and many more, and play a significant role in statistics and probability theory.

Ordinary moment

Moments and incomplete moments are significant because they can reveal details about the characteristics of a probability distribution. They can be used to determine the mean, variance, skewness, and kurtosis of a distribution, among other statistical measures. Moments and incomplete moments are also employed in the development and evaluation of statistical models. Moments are used in physics to characterize the geographical and temporal distributions of particles, as well as in finance to simulate stock price changes and determine risk factors. The n^{th} ordinary moment of w is given by

$$\mu_n' = E(W^n) = \sum_{\hbar=0}^{\infty} C_{[\hbar]} \int_0^{\infty} w^n \, h_{[a,b(1+\hbar)]}(w) dw.$$

or

$$\mu_n' = E(W^n) = \sum_{\hbar=0}^{\infty} C_{[\hbar]} b(1+\hbar) B\left(b(1+\hbar) - \frac{n}{a}, \frac{a+n}{a}\right)|_{(n<ab(1+\hbar))}. \quad (8)$$

Setting $n = 1$ in (8), we have the mean of W. The s^{th} central moment and cumulants (κ_s) of W are easily to be derived. The effects of the parameters α, β, a and b for the GOLLBXII model on the mean (μ_1'), variance (V(W)), skewness (S(W)) and kurtosis (K(W)) are listed in Table 2. The effects of the parameters a and b for the standard BXII model on the μ_1' , V(W), S(W) and K(W) are listed in Table 3. A useful comment has been added below.

Table 2: Numerical results for μ_1', V(W), S(W), K(W) for the GOLLBXII model.

α	β	a	b	μ_1'	V(W)	S(W)	K(W)
1.95	2	1.5	1.5	1.302554	0.43853	3.681042	95.18866
2				1.295751	0.40702	3.471017	72.59238
5				1.190338	0.04465	0.947154	6.448482
10				1.175915	0.01058	0.455053	4.701262
20				1.172331	0.00261	0.225332	4.321984
30				1.171668	0.00116	0.161642	3.855605
40				1.171436	0.00065	0.114757	4.122217
100				1.171185	0.00010	0.044932	4.204267
200				1.171150	2.59×10^{-5}	0.022595	4.170502
5	0.05	3	3	0.011348	0.0002338	5.076655	52.9221
	0.10			0.077045	0.0019812	1.614169	8.21755
	0.50			0.466824	0.0045673	0.230032	4.07226
	1			0.639627	0.0038438	0.217046	4.16290

	5			0.994109	0.0029583	0.326761	4.33864
	50			1.471921	0.0031319	0.4315871	4.52236
	100			1.622678	0.0033858	0.4494145	4.55841
	200			1.780042	0.0037238	0.463127	4.587191
	500			2.000840	0.0043072	0.4765337	4.616258
	1000			2.179385	0.0048623	0.4840242	4.632928
10	10	0.1	5	0.0437372	0.0012948	26.46451	3011661
		0.5		0.5184903	0.0040130	0.642670	5.20959
		1		0.7187439	0.0018975	0.335235	4.44217
		5		0.9358105	0.0001278	0.102164	4.19487
		50		0.9933281	5.380×10^{-5}	-131.7976	17848.4
		75		0.9955293	5.324×10^{-5}	-134.7817	18388.7
2	1.5	2.5	1	1.318956	0.2296088	2.653981	32.90810
			5	0.552977	0.0146199	0.495451	4.107098
			10	0.408876	0.0071186	0.343220	3.745968
			20	0.210700	0.0017257	0.228867	3.544562
			50	0.159300	0.0009754	0.214978	3.523984
			100	0.159303	0.0009754	0.214979	3.523984
			150	0.135346	0.0007014	0.210368	3.517327
			200	0.120587	0.0005558	0.208066	3.514019
			500	0.083525	0.0002657	0.203928	-181.8733
			750	0.071009	0.0001919	0.203010	24.35556
			1000	0.063285	0.0001524	0.202550	310.5896

Table 3: Numerical results for μ_1', V(W), S(W), K(W) for the BXII model.

a	b	μ_1'	V(W)	S(W)	K(W)
0.5	10	0.0277778	0.0071648	18.60987	2139.368
1		0.1111111	0.0154321	2.811057	17.82857
5		0.5864861	0.0196053	-0.1148575	2.927999
10		0.7599368	0.0089821	-0.5187003	3.462077
15		0.8312523	0.0049489	-0.6735949	3.837231
30		0.9108827	0.0015449	-0.8436482	4.367737
50		0.9453193	0.0006092	-0.9169383	4.637870
100		0.9721914	0.0001632	-0.9741929	4.866479
5	1	1.0689590	0.1786323	2.4852760	29.55619
	5	0.6824240	0.0289950	0.0401489	3.070043
	10	0.5864861	0.0196052	-0.1148575	2.927999
	20	0.5074029	0.0140696	-0.1862726	2.894430
	50	0.4208969	0.0094457	-0.2273829	2.883834
	100	0.3659695	0.0070836	-0.2408126	2.881726
	150	0.3373279	0.0060021	-0.2452596	2.881174
	200	0.3184032	0.0053403	-0.2474775	2.880926
	500	0.264992	0.0036900	-0.2514607	2.880525
	1000	0.2306612	0.0027936	-0.2527858	2.880246
	2000	0.2007902	0.002116076	-0.2534511	2.880363
	10000	0.1455217	0.001111123	-0.2539773	2.880306
	50000	0.1054701	0.000583630	-0.2540836	2.880286
	100000	0.0918169	0.000442305	-0.254098	2.880301

Form Tables 2 and we note that:

1-The new additional shape parameters α and β has an effect on μ_1' , V (W) , S (W) and K (W).

2-For the GOLLBXII model, S (W) can range in the interval ($-$ 134.7817, 26.46451). However, for the BXII model, S (W) can range in the interval ($-$ 0.974193, 18.60987).

3-For the GOLLBXII model, K (W) can range in the interval ($-$ 181.8733, 3011661). However, for the BXII model, K (W) can range in the interval (2.880286, 2139.368).

Moment generating function

In probability theory and statistics, the moment generating function (MGF) is a crucial mathematical operation. It offers a potent tool for examining the characteristics of probability distributions and drawing conclusions from statistics. The expected value of an exponential function of a random variable is the definition of the moment generating function, which is used to determine the distribution's moments. The moment generating function is significant because it can reveal details about the characteristics of a probability distribution. The distribution's moments, including the mean, variance, skewness, and kurtosis, can be determined specifically using the moment generating function. The characteristic function and other significant statistical measures can also be derived from the moment generating function. The MGF of W, say $M_W(t) = E[exp(tW)]$, can be firstly obtained from (6) as

$$M_W(t) = \sum_{\hbar=0}^{\infty} C_{[\hbar]} M_{[a,b(1+\hbar)]}(t),$$

where $M_{[a,b(1+\hbar)]}(t)$ is the MGF of the BXII distribution with parameters a and $b(1+\hbar)$. Next, consider the Meijer **G** (Mj-**G**) function (see Gradshteyn and Ryzhik (2000)) which is defined by

$$G_{p,q}^{m_1,n}\left(w\Big|{a_1, a_2, \dots, a_p \atop b_1, b_2 \dots, b_q}\right) = \frac{1}{2\pi i}\int_{I_{[G]}} \frac{\prod_{\hbar=1}^{m_1} \Gamma(b_\hbar + t)\prod_{\hbar=1}^{n} \Gamma(1-a_\hbar - t)}{\prod_{\hbar=n+1}^{p} \Gamma(a_\hbar + t)\prod_{\hbar=m_1+1}^{p} \Gamma(1-b_\hbar - t)} w^{-t} dt,$$

where $i = (-1)^{\frac{1}{2}}$ is the complex unit and $I_{[G]}$ denotes an integration path. According to Prudnikov et al. (1986) and Prudnikov et al. (1992), the Mj-**G** function contains many integrals with elementary and special functions as particular cases. Assume that $a = m_1/m_2$, where m_1 and m_2 are positive integers. We have the following result

$$I\left(p,\mu,\frac{m_1}{m_2},v\right)\Big|_0^\infty = \int_0^\infty \frac{e^{-pw} w^{\mu}}{\left(1+w^{\frac{m_1}{b}}\right)^{-v}} dw = K_{(p,\mu,m_1,v)} G_{m_2+m_1,m_2}^{m_2,m_2+m_1},$$

where

$$G_{m_2+m_1,m_2}^{m_2,m_2+m_1} = G_{m_2+m_1,m_2}^{m_2,m_2+m_1}\left(\left(\frac{m_1}{p}\right)^{m_1}\Big|{\Delta(m_1,-\mu),\Delta(m_2,1+v) \atop \Delta(m_2,0)}\right),$$

$$K_{(p,\mu,m_1,v)} = \frac{m_2^{-v} p^{-(\mu+1)} m_1^{\mu+\frac{1}{2}}}{\Gamma(-v)(2\pi)^{\frac{m_1-1}{2}} p^{\mu+1}}\Big|_{(\mu>-1 \text{ and } p>0)}$$

and

$$\Delta(\omega_1,\omega_2) = \frac{\omega_1}{\omega_2}, \frac{\omega_1+1}{\omega_2}, \frac{\omega_1+2}{\omega_2}, \dots, \frac{\omega_1+\omega_2}{\omega_2}.$$

Then, we can write

$$M(t) = m_1\, I\left(-t, \frac{m_1}{m_2}-1, \frac{m_1}{m_2}, -m_2-1\right)\Big|_{(t<0)}.$$

Hence, the MGF of W can be expressed as

$$M_W(t) = \hbar \sum_{\hbar=0}^{\infty} C_{[\hbar]} I\left(-t, \frac{\hbar}{\boldsymbol{b}(1+\hbar)} - 1, \frac{\hbar}{\boldsymbol{b}(1+\hbar)}, -[1+\boldsymbol{b}(1+\hbar)]\right).$$

Incomplete Moments

The $s^{t\hbar}$ incomplete moment, say $I_s(t)$, of the GOLLBXII distribution is given by $I_s(t) = \int_0^t w^s f(w) dw$. From equation (7), we have

$$I_s(t) = \sum_{\hbar=0}^{\infty} C_{[\hbar]} \int_0^t w^s \hbar_{[a,\boldsymbol{b}(1+\hbar)]}(w) dw,$$

and using the lower incomplete gamma function, we obtain

$$I_s(t) = \sum_{\hbar=0}^{\infty} C_{[\hbar]} \boldsymbol{b}(1+\hbar) B\left(t^{a}; \boldsymbol{b}(1+\hbar) - \frac{s}{\boldsymbol{a}}, \frac{\boldsymbol{a}+s}{\boldsymbol{a}}\right).$$

The first incomplete moment (FIM) of W, denoted by $I_1(t)$, is simply determined from the above equation by setting $s = 1$. The FIM has many economical applications related to the Lorenz and Bonferroni curves. When a machine or component is going to break down can be foreseen using the residual life function. Maintenance staff can predict when to replace or perform maintenance on a machine by tracking its state and computing its residual life function. Analysis of product warranties can be done using the residual life function. The function allows manufacturers to predict the likelihood that their devices will break during the warranty period and modify the guaranteed conditions as necessary. In medical research, survival data can be analyzed using the residual life function. It can be used, for instance, to calculate the likelihood that a patient will live a specific amount of time following a diagnostic or medical procedure.

Residual and reversed residual life functions

The complement of the residual life function is the reversed residual life function, commonly referred to as the exceedance probability function. Given that an object has so far survived, it describes the likelihood that it will fail before a specific time. The residual life function, often referred to as the remaining life function, is a function that is used in dependability theory to characterize the likelihood that a product will fail after a specific amount of time has passed, assuming that it has lasted up to that point. The conditional survival probability, assuming the object has already survived up to a particular point, can be defined as the residual life function. The n^{th} moment of the residual life (RL), denoted by $m_n(t) = E[(W-t)^n]|_{(W>t,\ n=1,2,\dots)}$. The n^{th} moment of the residual life of W is given by

$$m_n(t) = \frac{\int_t^{\infty} (W-t)^n f_{\underline{\Psi}}(w) dw}{1 - F_{\underline{\Psi}}(t)}.$$

Then, we can write

$$m_n(t) = \frac{1}{1 - F_{\underline{\Psi}}(t)} \sum_{i=0}^{n} \sum_{\hbar=0}^{\infty} C_{[\hbar]}^{*} \boldsymbol{b}(1+\hbar) B\left(t^{\alpha}; \boldsymbol{b}(1+\hbar) - \frac{n}{\boldsymbol{a}}, \frac{\boldsymbol{a}+n}{\boldsymbol{a}}\right),$$

where $C_{[\hbar]}^{*} = C_{[\hbar]} \frac{(-1)^{n-i} n! t^{n-i}}{i! \Gamma(n-i+1)}$. The n^{th} moment reversed residual life

$$M_n(t) = E[(t-W)^n]|_{(t>0, w\le t \text{ and } n=1,2,\dots)}$$

Then, $M_n(t)$ is defined by

$$M_n(t) = \frac{\int_0^t (t-W)^n f_{\underline{\Psi}}(w) dw}{F_{\underline{\Psi}}(t)}.$$

The n^{th} moment of the reversed residual life of W

$$M_n(t) = \frac{1}{F_{\underline{\Psi}}(t)} \sum_{i=0}^{n} \sum_{\hbar=0}^{\infty} C_{[\hbar]}^{**} \boldsymbol{b}(1+\hbar) B\left(t^{\alpha}; \boldsymbol{b}(1+\hbar) - \frac{n}{\boldsymbol{a}}, \frac{\boldsymbol{a}+n}{\boldsymbol{a}}\right),$$

where $C_{[\hbar]}^{**} = C_{[\hbar]} \frac{(-1)^{i} n!}{i!(n-i)!}$. Reliability engineering frequently examines the dependability of complicated systems using the reversed residual life function. Given that a system component has already been in operation for a while, it can be used to estimate the likelihood that it will fail before a particular time. In risk management, the reversed residual life function can be used to calculate the likelihood that an event will occur within a specific time period. This can be used to evaluate the risk involved in a specific investment or activity. In quality control, the reversed residual life function can be used to calculate the likelihood that a fault will manifest itself within a given time frame. This can be used to establish quality control requirements and confirm that the items meet those requirements.

3. Copula

Copula is an important statistical concept used in modeling bivariate or multivariate data. It is a function that links the marginal distributions of two or more variables to their joint distribution. The use of copulas has become increasingly popular in recent years due to their flexibility and ability to model complex dependence structures between variables. Here are some of the key importance and usage of copulas in statistics and bivariate data modeling. Copulas are used to model dependence between variables, which is an important concept in many areas of statistics and data science. Copulas allow us to model the joint distribution of variables while retaining the marginal distributions of each variable, making it easier to capture complex dependencies between variables that cannot be captured by simple correlation measures. Copulas are used in finance to model the dependence structure between asset returns. This is important in portfolio optimization, where the aim is to construct a portfolio of assets that maximizes returns while minimizing risk. By using copulas to model the dependence between assets, we can better estimate the risk of a portfolio and construct more efficient portfolios. Copulas are also used in risk management, where they can be used to estimate the risk of extreme events, such as market crashes or natural disasters. By modeling the dependence structure between variables, copulas can provide a more accurate estimate of the probability of such events occurring, which is important for risk management and insurance. Copulas are also used for data generation, which is useful in situations where data is scarce or expensive to collect. By modeling the dependence structure between variables using a copula, we can simulate new data sets that have similar dependence structures as the original data, allowing us to generate new data for analysis and testing. In summary, copulas are an important statistical concept used in modeling bivariate or multivariate data. They are used to model dependence between variables, which is important in many areas of statistics and data science, including portfolio optimization, risk management, and data generation. By using copulas, we can capture complex dependencies between variables and make more accurate estimates of risk and other important parameters.

Bivariate GOLLBXII type distribution via Morgenstern family

First, we start with CDF for Morgenstern family of two random variables W_1 and W_2 which can be written as

$$F_\lambda(w_1,w_2)|_{(|\lambda|\le 1)} = F_1(w_1)F_2(w_2)\{1+\lambda[1-F_1(w_1)][1-F_2(w_2)]\},$$

setting

$$F_{\underline{\Psi}_1}(w_1) = \frac{\left[1-\left(1+w_1^{a_1}\right)^{-b_1}\right]^{\alpha_1\beta_1}}{\left[1-\left(1+w_1^{a_1}\right)^{-b_1}\right]^{\alpha_1\beta_1} + \left\{1-\left[1-\left(1+w_1^{a_1}\right)^{-b_1}\right]^{\beta_1}\right\}^{\alpha_1}},$$

where $\underline{\Psi}_1 = (\alpha_1, \beta_1, \boldsymbol{a}_1, \boldsymbol{b}_1)$ and

$$F_{\underline{\Psi}_2}(w_2) = \frac{\left[1-\left(1+w_2^{a_2}\right)^{-b_2}\right]^{\alpha_2\beta_2}}{\left[1-\left(1+w_2^{a_2}\right)^{-b_2}\right]^{\alpha_2\beta_2} + \left\{1-\left[1-\left(1+w_2^{a_2}\right)^{-b_2}\right]^{\beta_2}\right\}^{\alpha_2}},$$

where $\underline{\Psi}_2 = (\alpha_2, \beta_2, \boldsymbol{a}_2, \boldsymbol{b}_2)$ then we have a 9-dimension parameter model as

$$F(w_1,w_2) = \frac{\left[1-\left(1+w_1^{a_1}\right)^{-b_1}\right]^{\alpha_1\beta_1}}{\left[1-\left(1+w_1^{a_1}\right)^{-b_1}\right]^{\alpha_1\beta_1} + \left\{1-\left[1-\left(1+w_1^{a_1}\right)^{-b_1}\right]^{\beta_1}\right\}^{\alpha_1}}$$
$$\times \frac{\left[1-\left(1+w_2^{a_2}\right)^{-b_2}\right]^{\alpha_2\beta_2}}{\left[1-\left(1+w_2^{a_2}\right)^{-b_2}\right]^{\alpha_2\beta_2} + \left\{1-\left[1-\left(1+w_2^{a_2}\right)^{-b_2}\right]^{\beta_2}\right\}^{\alpha_2}}$$

$$\times\left\{1+\lambda\left(\begin{array}{c}\left\{1-\frac{\left[1-\left(1+w_1^{a_1}\right)^{-b_1}\right]^{\alpha_1\beta_1}}{\left[1-\left(1+w_1^{a_1}\right)^{-b_1}\right]^{\alpha_1\beta_1}+\left\{1-\left[1-\left(1+w_1^{a_1}\right)^{-b_1}\right]^{\beta_1}\right\}^{\alpha_1}}\right\}\\ \times\left\{1-\frac{\left[1-\left(1+w_2^{a_2}\right)^{-b_2}\right]^{\alpha_2\beta_2}}{\left[1-\left(1+w_2^{a_2}\right)^{-b_2}\right]^{\alpha_2\beta_2}+\left\{1-\left[1-\left(1+w_2^{a_2}\right)^{-b_2}\right]^{\beta_2}\right\}^{\alpha_2}}\right\}\end{array}\right)\right\}.$$

Bivariate GOLLBXII type distribution via clayton copula

The bivariate extension via Clayton Copula can be considered as a weighted version of the Clayton copula, which is of the form

$$C(u,v)=\left[u^{-(\phi_1+\phi_2)}+v^{-(\phi_1+\phi_2)}-1\right]^{-\frac{1}{\phi_1+\phi_2}}.$$

This is indeed a valid copula. Next, let us assume that $W\sim$ GOLLBXII$(\alpha_1,\beta_1,\boldsymbol{a}_1,\boldsymbol{b}_1)$ and $Y\sim$ GOLLBXII$(\alpha_2,\beta_2,\boldsymbol{a}_2,\boldsymbol{b}_2)$. Then, setting

$$u=u_{\underline{\Psi}_1}(w)=\frac{\left[1-(1+w^{a_1})^{-b_1}\right]^{\alpha_1\beta_1}}{[1-(1+w^{a_1})^{-b_1}]^{\alpha_1\beta_1}+\{1-[1-(1+w^{a_1})^{-b_1}]^{\beta_1}\}^{\alpha_1}}$$

and

$$v=v_{\underline{\Psi}_2}(y)=\frac{\left[1-(1+y^{a_2})^{-b_2}\right]^{\alpha_2\beta_2}}{[1-(1+y^{a_2})^{-b_2}]^{\alpha_2\beta_2}+\{1-[1-(1+y^{a_2})^{-b_2}]^{\beta_2}\}^{\alpha_2}},$$

the associated CDF bivariate GOLLBXII type distribution will be

$$C(x,y)=\left(\begin{array}{c}\left\{\frac{\left[1-(1+w^{a_1})^{-b_1}\right]^{\alpha_1\beta_1}}{[1-(1+w^{a_1})^{-b_1}]^{\alpha_1\beta_1}+\{1-[1-(1+w^{a_1})^{-b_1}]^{\beta_1}\}^{\alpha_1}}\right\}^{-(\phi_1+\phi_2)}\\ +\left\{\frac{\left[1-(1+y^{a_2})^{-b_2}\right]^{\alpha_2\beta_2}}{[1-(1+y^{a_2})^{-b_2}]^{\alpha_2\beta_2}+\{1-[1-(1+y^{a_2})^{-b_2}]^{\beta_2}\}^{\alpha_2}}\right\}^{-(\phi_1+\phi_2)}\\ -1\end{array}\right)^{-\frac{1}{\phi_1+\phi_2}}.$$

Note: Depending on the specific baseline CDF, one may construct various bivariate GOLLBXII type model in which $(\phi_1+\phi_2)\geq 0$.

Multivariate GOLLBXII type distribution via Clayton copula

A straightforward d-dimensional extension from the above will be

$$H(x_1,x_2,\cdots,x_d)=\left[\sum_{i=1}^{d}\left(\frac{\left[1-\left(1+w_i^{a_i}\right)^{-b_i}\right]^{\alpha_i\beta_i}}{\left\{\left[1-\left(1+w_i^{a_1}\right)^{-b_i}\right]^{\alpha_i\beta_i}+\left\{1-\left[1-\left(1+w_i^{a_i}\right)^{-b_i}\right]^{\beta_i}\right\}^{\alpha_i}\right\}}\right)^{-(\phi_1+\phi_2)}+1-d\right]^{-1/(\phi_1+\phi_2)}.$$

4.Estimation

4.1 Maximum likelihood method

Suppose that $(w_1,\cdots,w_n)$ is a random sample (rs) from the GOLLBXII model. The log-likelihood function ($\ell_n(\underline{\Psi})$) for $\underline{\Psi}$ is given by

$$\ell_n(\underline{\Psi})=n\,log(\alpha\beta\boldsymbol{a}\boldsymbol{b})+(\boldsymbol{a}-1)\sum_{i=1}^{n}log\,w_i-(\boldsymbol{b}+1)\sum_{i=1}^{n}log\,\Upsilon_i$$
$$+(\alpha\beta-1)\sum_{i=1}^{n}log\left(1-\Upsilon_i^{-\boldsymbol{b}}\right)+(\alpha-1)\sum_{i=1}^{n}log\left[1-\left(1-\Upsilon_i^{-\boldsymbol{b}}\right)^{\beta}\right]$$

$$-2\sum_{i=1}^{n} \quad log\left\{\left(1-\Upsilon_i^{-\mathcal{b}}\right)^{\alpha\beta}+\left[1-\left(1-\Upsilon_i^{-\mathcal{b}}\right)^{\beta}\right]^{\alpha}\right\}$$

where $\Upsilon_i=(w_i^{\mathcal{a}}+1)$. The above $\ell_n(\underline{\Psi})$ can be maximized by numerical methods via many programs such as SAS (PROC NLMIXED) or R (optim) or Ox program (via sub-routine MaxBFGS). The components of the score vector

$I_{(\underline{\Psi})}=\frac{\partial\ell}{\partial\underline{\Psi}}=\left(\frac{\partial\ell_n(\underline{\Psi})}{\partial\alpha},\frac{\partial\ell_n(\underline{\Psi})}{\partial\beta},\frac{\partial\ell_n(\underline{\Psi})}{\partial\mathcal{a}},\frac{\partial\ell_n(\underline{\Psi})}{\partial\mathcal{b}}\right)^T$ can be easily derived.

4.2 Simulations

Simulation studies are an essential tool in statistical analysis for several reasons:

I. Assessing the validity of statistical methods: Simulation studies allow statisticians to evaluate the performance of different statistical methods by generating data under specific conditions and testing the accuracy of the methods in estimating the true parameters. This provides a means of assessing the validity of the methods and their suitability for different types of data.

II. Testing hypotheses: Simulation studies can be used to test hypotheses by generating data that reflect the null or alternative hypothesis and comparing the results obtained from the simulated data with the observed data. This allows researchers to determine the statistical significance of their findings and assess the power of their tests.

III. Sample size determination: Simulation studies can be used to determine the appropriate sample size required for a study. By generating data with different sample sizes and assessing the performance of different statistical methods, researchers can determine the sample size that will yield the most accurate and reliable results.

IV. Sensitivity analysis: Simulation studies can be used to assess the sensitivity of statistical methods to different assumptions and parameters. This helps researchers to identify potential biases and assess the robustness of their results.

V. Teaching and learning: Simulation studies can be used as a teaching tool to help students understand statistical concepts and methods. By generating data and manipulating different parameters, students can develop an intuitive understanding of statistical concepts and gain hands-on experience with statistical software. Overall, simulation studies play a crucial role in statistical analysis by providing a means of evaluating the performance of statistical methods, testing hypotheses, determining sample sizes, assessing the sensitivity of methods, and facilitating learning and teaching.

There are several types of simulation studies that can be conducted in statistical analysis, including:

I. Monte Carlo simulation: Monte Carlo simulation involves generating random samples from a population and analyzing the properties of the samples using statistical methods. This type of simulation is often used to estimate the properties of complex systems or models.

II. Bootstrap simulation: Bootstrap simulation involves generating many samples by resampling from a single sample, with replacement. This technique is often used to estimate the variability of a statistic or to construct confidence intervals.

III. Permutation testing: Permutation testing involves permuting the data in order to test hypotheses about the population. This technique is often used when the assumptions underlying traditional statistical tests are not met.

IV. Sensitivity analysis: Sensitivity analysis involves systematically varying one or more parameters in a statistical model or simulation to assess the impact on the results. This type of simulation is often used to assess the robustness of statistical methods.

V. Power analysis: Power analysis involves simulating data with known effect sizes to determine the probability of detecting a significant effect. This technique is often used in sample size determination and to plan statistical studies.

VI. Agent-based simulation: Agent-based simulation involves modeling the behavior of individuals or agents in a system and analyzing the emergent properties of the system. This type of simulation is often used in social sciences, economics, and ecology. Overall, simulation studies are a powerful tool in statistical analysis, and the choice of simulation method will depend on the research question, the type of data, and the statistical methods being used.

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Graphically, we can perform the simulation experiments to assess of the finite sample behavior of the MLEs. The assessment was based on the following algorithm:

- Use

$$w_u = \left[\left(1 - \left[\frac{\left(\frac{u}{1-u}\right)^{\frac{1}{\alpha}}}{1 + \left(\frac{u}{1-u}\right)^{\frac{1}{\alpha}}} \right]^{\frac{1}{\beta}} \right)^{-\frac{1}{\boldsymbol{b}}} - 1 \right]^{\frac{1}{\boldsymbol{a}}}$$

- to generate 1000 samples of size n from the GOLLBXII distribution.
- Compute the MLEs for the 1000 samples.
- Compute the SEs of the MLEs for the 1000 samples.
- the standard errors (SEs) were computed by inverting the observed information matrix.
- compute the biases and mean squared errors given for $\hbar = \alpha, \beta, \boldsymbol{a}, \boldsymbol{b}$. We repeated these steps for $n = 50,100, \dots ,500$ with $\alpha = 1,\ \beta = 1,\ \boldsymbol{a} = 1,\ \boldsymbol{b} = 1$ so computing biases ($B_{\hbar}(n)$), mean squared errors ($MSEs$) $(MSE_{\hbar}(n))$ for $\alpha, \beta, \boldsymbol{a}, \boldsymbol{b}$ and $n = 50,100, \dots ,500$.

Figure 2 (left panel) shows how the four biases vary with respect to n . Figure 2 (right panel) shows how the four MSEs vary with respect to n . The broken lines in Figure 2 corresponds to the biases being 0 . From Figure 2, the biases for each parameter decrease to zero as $n \to \infty$, the MSEs for each parameter decrease to zero as $n \to \infty$.

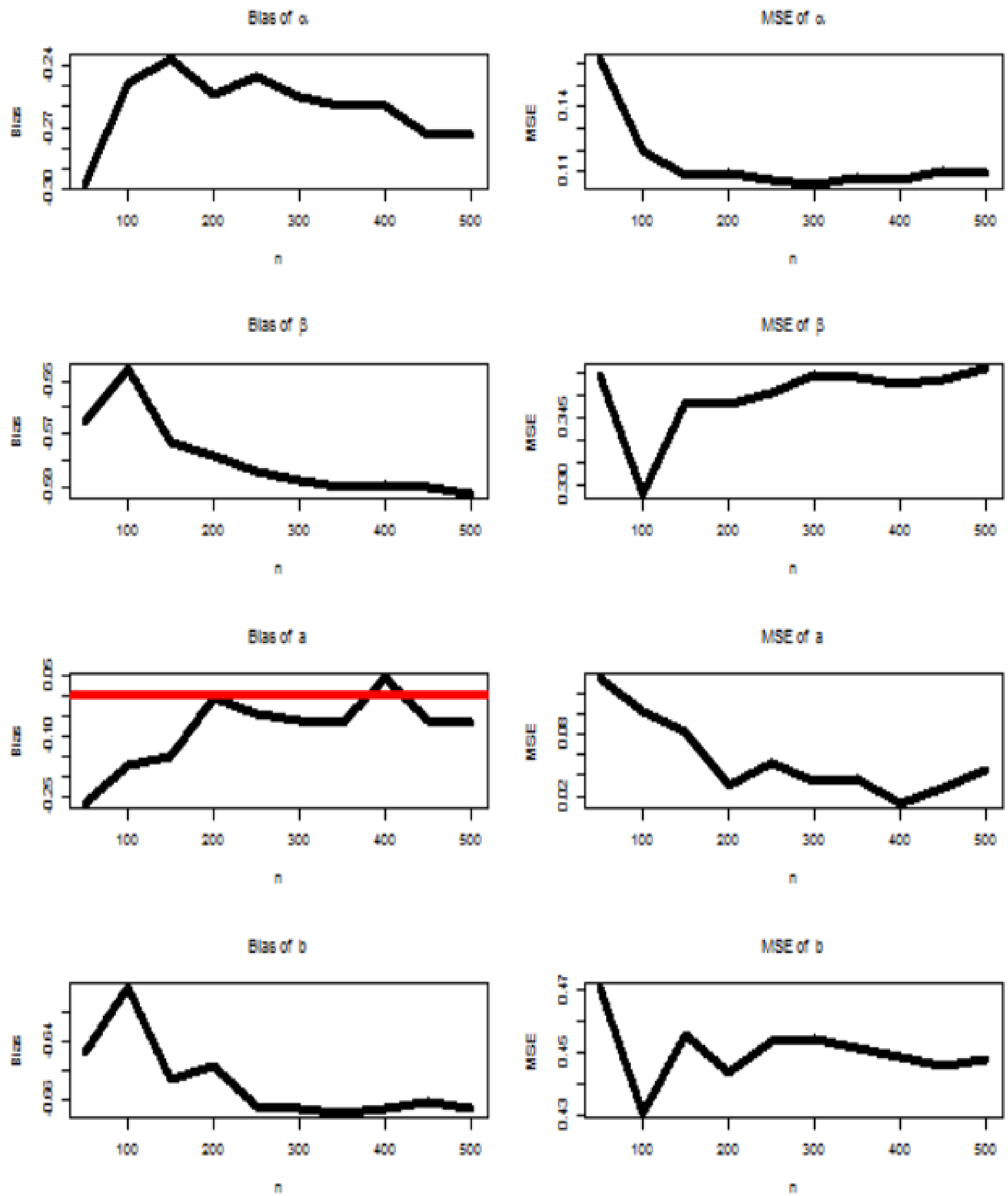

Figure 2: Biases and MSEs for ; ; a; b and n = 50; 100; : : : ; 500 for the new model.

5.Three applications for comparing models

A probability distribution's adaptability relates to its capacity to represent many kinds of data structures and patterns. The normal distribution, the Poisson distribution, the exponential distribution, and the gamma distribution are a few of the frequently used probability distributions. Each of these distributions has distinct qualities and advantages that make it appropriate for representing kinds of data. In order to assess whether a new probability distribution is appropriate for the task at hand when investigating its flexibility, it is crucial to take into account the characteristics of the data being modelled and to compare the new distribution to other widely used distributions. A distribution's adaptability can also be increased by mixing different distributions in a mixture model or by modelling non-typical data patterns with transformations. A symmetrical distribution that works well for modelling data with a bell-shaped pattern is the normal distribution, for instance. The Poisson distribution, on the other hand, is used to model data that shows the number of events that occur over a specified period of time or place. For the purpose of modelling data that illustrates the interval between events in a Poisson process, the exponential distribution is used. To illustrate the flexibility of the GOLLBXII model, we provide three applications. Data set I called the data of breaking stress which consists of 100 observations and given by Nichols and Padgett (2006). Data set II called survival times in days of 72 guinea pigs infected with virulent tubercle bacilli, originally observed and reported by Bjerkedal, T. (1960). Data set III called taxes revenue data in 1000 million Egyptian pounds given in Altun et al. (2018 a, b). Many useful data sets which can be modeled under the new model are available in Elsayed and Yousof, H. M. (2019a, 2019b, 2020 and 2021), Salah et al. (2022) and Elgohari and Yousof (2020a, 2020b, 2020c and 2021).

For all data sets, we compare the GOLLBXII distribution, with BXII, Marshall-Olkin BXII (MOBXII), Topp-Leone BXII (TLBXII), Zografos-Balakrishnan BXII (ZBTBXII), five parameters beta-BXII (FPBTBXII), BTBXII, Beta-exponentiated-BXII (BEBXII), Five Parameters Kumaraswamy-BXII (FKwmBXII) and KwmBXII distributions. All competitive models are given in Yousof et al. (2017), Yousof et al. (2018 a, b), Alizadeh et al. (2018), Korkmaz et al. (2018 a, b), Altun et al. (2018 a, b), Alizadeh et al. (2019) and Yousof et al. (2019 a, b). We consider the well-known G-O-F statistics: the Akaike Information Criterion $\left(C_{[\mathrm{AI}]}\right)$, Bayesian Information Criterion $\left(C_{[\mathrm{Bayes}]}\right)$, Hannan-Quinn Information Criterion $\left(C_{[\mathrm{HQ}]}\right)$, Consistent Akaike Information Criterion $\left(C_{[\mathrm{CA}]}\right)$.

Tables 4, 5, 6 give the MLEs, standard errors (SEs), confidence interval (CL) for the data set **I, II, III** respectively and give also the $C_{[\mathrm{AI}]}$, $C_{[\mathrm{Bayes}]}$, $C_{[\mathrm{HQ}]}$ and $C_{[\mathrm{CA}]}$ values for the data set **I, II, III** respectively**.** Figure 3 gives the box plots. Figure 4 gives the Quantile-Quantile plots. Based on Figure 3 and Figure 4, it is seen that the three data sets have extreme values. Extreme value datasets often occur in many fields such as meteorology, hydrology, finance, and engineering. Modeling these datasets using probability distributions can help us understand and quantify the behavior of extreme events and make predictions for future extreme events. Here are three common probability distributions used to model extreme value datasets ؛The Generalized Extreme Value (GEV) Distribution: The GEV distribution is often used to model the maximum or minimum values in a dataset. It has three parameters: location, scale, and shape. The GEV distribution can be used to model data from a wide range of fields, including meteorology, finance, and engineering. When modeling extreme value datasets, it is important to choose the appropriate distribution that fits the data well. This can be done by using statistical tests and visualizations to compare the data to the distribution. Once a suitable distribution is found, it can be used to make predictions for future extreme events and assess the risk associated with these events. Figures 5, 6, 7 give the TTT plot, P-P plot, E.P.D.F, E.C.D.F, E.H.R.F and Kaplan-Meier survival plots for the data set **I, II, III** respectively**.** Based on the values in Tables 4, 5, 6 and Figures 5, 6, 7 the GOLLBXII model has the best fits as compared to BXII other models in the three applications with small values for $C_{[\mathrm{AI}]}$, $C_{[\mathrm{Bayes}]}$, $C_{[\mathrm{HQ}]}$ and $C_{[\mathrm{CA}]}$.

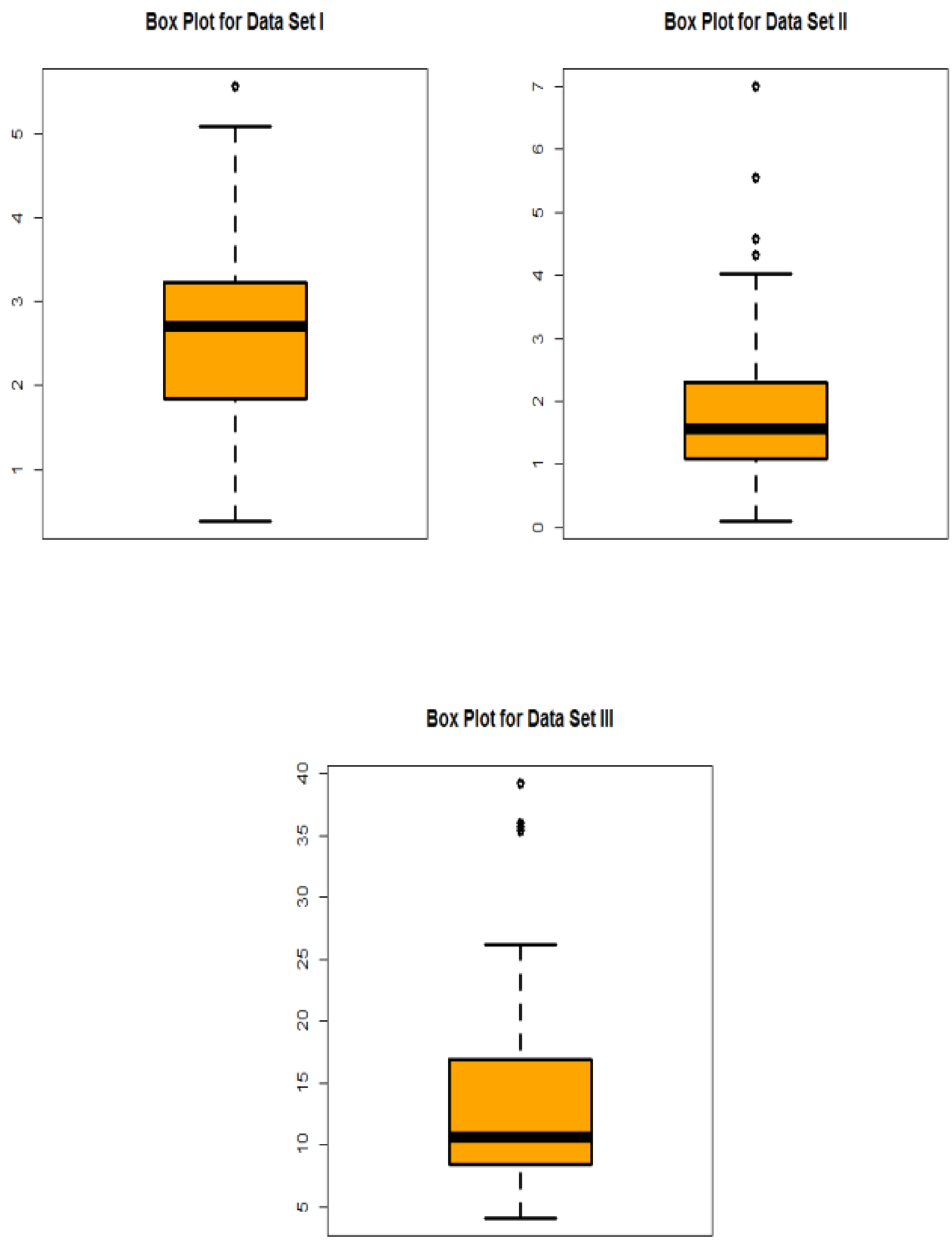

Figure 3: Box plots.

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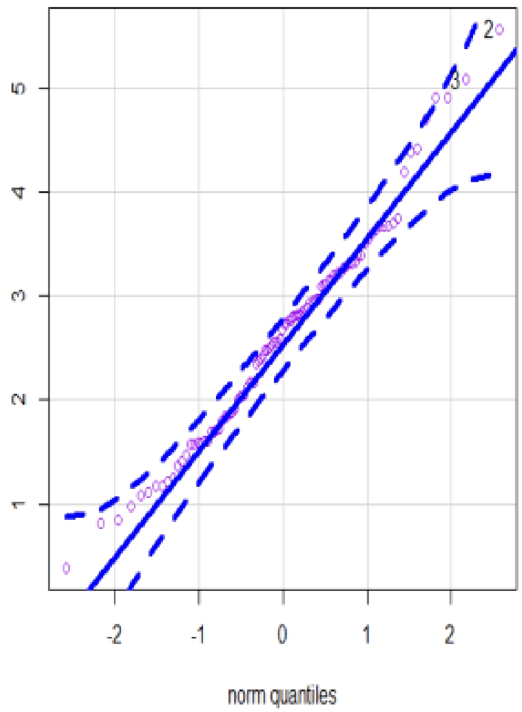
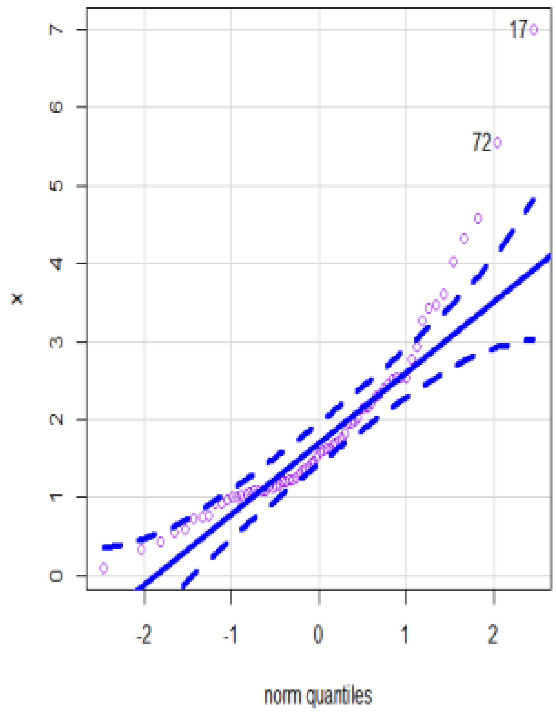


Data set I.



Data set II.

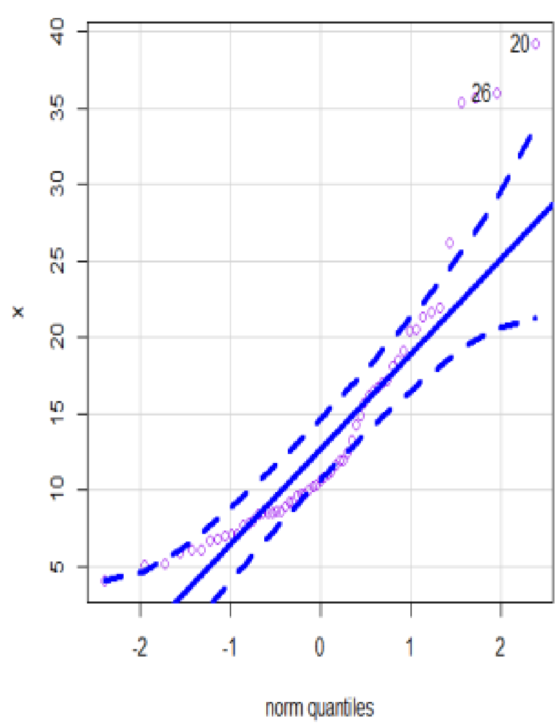


Figure 4: Q-Q plots.

Table 4: MLEs, SEs, C_{AI} , C_{Bayes} , C_{HQ} and C_{CA} for the data set I.

Model	$\hat{\alpha},\hat{\beta},\hat{\boldsymbol{a}},\hat{\boldsymbol{b}},\hat{\gamma}$	C_{AI} , C_{Bayes} , C_{HQ} and C_{CA}
BXII	---, ---, 5.9413, 0.1874, ---	382.96, 388.16, 383.10, 385.10
	---, ---, (1.2792), (0.0441), ---	
	---, ---, (3.431,8.452), (0.101,0.270), ---	
MOBXII	---, ---, 1.1922,4.8343,838.734	305.9 313.60, 306.02, 308.97
	---, ---, (0.9523), (4.8960),(229.342)	
	---, ---, (0, 3.060), (0,14.431),(389.221,1288.23)	
TLBXII	---, ---, 1.3501,1.0612,13.7282	323.54, 331.40, 323.78, 326.72
	---, ---, (0.3781), (0.3842) ,(8.403)	
	---, ---, (0.613, 2.090), (0.310,1.801) ,(0, 30.191)	
KwmBXII	48.105 ,79.514 ,0.352 ,2.738, ---	303.78, 314.23, 304.20, 308.02
	(19.344),(58.182) ,(0.0981) ,(1.0771) ,---	
	(10.17,86.03), (0,193.54) ,(0.15,0.54), (0.60,4.86),---	
BTBXII	359.683 ,260.097 ,0.175 ,1.123, ---	305.65, 316.16, 306.26, 309.86
	(57.943),(132.223),(0.015),(0.244),---	
	(246,473.5), (0.95,519.5), (0.14,0.23), (0.65,1.66), ---	
BEBXII	0.3813, 11.9493, 0.9371, 33.406, 1.7057	305.83, 318.85, 306.46, 311.29
	(0.0781), (4.6351), (0.268), (6.288), (0.47815)	
	(0.2,0.55), (2.8,22), (0.4,1.8), (20,46), (0.81,2.69)	
FKwmBXII	0.54243,4.2231, 5.316, 0.415, 4.1526	305.52, 318.55, 306.44, 310.84
	(0.135), (1.881), (2.320), (0.4971), (1.9951)	
	(0.2, 0.9), (0.5,7.99), (0.91,9), (0, 1.8), (0.2,8.8)	
ZBTBXII	123.13, ---,0.3682, 139.2471, ---	302.98, 310.79, 303.24, 306.15
	(243.2), ---, (0.342), (318.55), ---	
	(0, 599), ---, (0, 1.05), (0, 763.6),---	
GOLLBXII	**10.778, 4.923 0.181, 2.605, ---**	**301.11, 310.53, 301.53, 305.33**
	11.251, 5.237, 0.227, 1.509, ---	
	(0, 33), (0, 5.2), (0, 0.227), (0, 5.6), ---	

Table 5: MLEs, SEs, C_{AI} , C_{Bayes} , C_{HQ} and C_{CA} for the data set II.

Model	$\hat{\alpha},\hat{\beta},\hat{\boldsymbol{a}},\hat{\boldsymbol{b}},\hat{\gamma}$	C_{AI} , C_{Bayes} , C_{HQ} and C_{CA}
BXII	---, ---, 3.1025, 0.4652, ---	209.66, 214.25, 209.78, 211.41
	---, ---, (0.537), (0.076), ---	
	---, ---, (2.051,4.162), (0.301,0.623), ---	
MO BXII	---, ---, 2.2591,1.5334, 6.763	209.75, 216.57, 210.10, 212.45
	---, ---, (0.861), (0.905), (4.585)	
	---, ---, (0.51,3.958), (0,3.315), (0, 15.76)	
TL BXII	---, ---, 2.391,0.4581,1.7962	211.81, 218.62, 212.16, 214.53
	---, ---, (0.901), (0.2444), (0.9155)	
	---, ---, (0.61,4.18), (0, 0.945), (0.001,3.60)	
KwmBXII	14.1054,7.4244, 0.5251, 2.2749, ---	208.77, 217.87, 209.38, 212.40
	(10.81), (11.852), (0.280), (0.991), ---	
	(0, 35.3), (0.30.8), (0, 1.1), (0.31, 4.25), ---	
FPBTBXII	0.624, 0.5491,3.84, 1.383, 1.6651	206.83, 218.22, 207.73, 211.35
	(0.543), (1.014), (2.782), (2.34), (0.44)	
	(0, 1.73), (0, 2.52), (0, 9.31), (0, 5.94), (0.7, 4.51)	
FKwmBXII	0.5573, 0.318, 3.998, 2.133, 1.4755	206.52, 217.91, 207.43, 211.22
	(0.440), (0.3141), (2.083), (1.8331), (0.360)	
	(0, 1.44), (0, 0.95), (0, 3.13), (0, 5.71), (0.75, 2.25)	
GOLLBXII	**6.506, 1.916, 0.319, 1.557, ---**	**205.72, 214.83, 206.32, 209.35**

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	12.963, 2.259, 0.781, 1.565, ---	
	(0, 32.5), (0, 6.5), (0, 4.5), (0, 4.8), ---	

Table 6: MLEs, SEs, C_{AI} , C_{Bayes} , C_{HQ} and C_{CA} for the data set III.

Model	$\hat{\alpha},\hat{\beta},\hat{\boldsymbol{a}},\hat{\boldsymbol{b}},\hat{\gamma}$	C_{AI} , C_{Bayes} , C_{HQ} and C_{CA}
BXII	---, ---, 5.617, 0.0724, ---	518.47, 522.61, 518.68, 520.10
	---, ---, (15.051), (0.1943), ---	
	---, ---, (0, 35.14), (0, 0.451), ---	
MOBXII	---, ---, 8.0171, 0.420, 70.360	387.23, 389.40, 387.67, 389.70
	---, ---, (22.080), (0.313), (63.833)	
	---, ---, (0, 51.30), (0, 1.031), (0, 195.474)	
TLBXII	---, ---, 91.323, 0.0124, 141.0734	385.93, 392.20, 386.41, 388.41
	---, ---, (15.074), (0.0022), (70.0283)	
	---, ---, (61.8,120.88) (0.007, 0.024) (3.80,278.35)	
KwmBXII	18.133, 6.87, 10.692, 0.0813, ---	385.60, 393.93, 386.33, 388.87
	(3.6890), (1.036), (1.167), (0.0122), ---	
	(10.8,25.4), (4.81,8.9), (8.4,12.988), (0.06,0.105), ---	
BTBXII	26.7251, 9.7562, 27.3641, 0.0203, ---	385.57, 394.13, 386.34, 389.14
	(9.46), (2.782), (12.353), (0.008), ---	
	(8.15,45.28), (4.3,15.23), (3.16,51.59), (0.005,0.037), ---	
BEBXII	2.9242, 2.9111, 3.278, 12.488, 0.3713	387.05, 397.44, 388.18, 391.10
	(0.565), (0.5491), (1.255), (6.9384), (0.7881)	
	(1.8,4.1), (1.8,3.9), (0.80,5.7), (0, 26.1), (0, 1.93)	
FP	30.4414, 0.585, 1.0891, 5.1666, 7.8621	386.75, 397.15, 387.90, 390.80
	(91.75), (1.061), (1.024), (8.27), (15.04)	
	(0, 210.3), (0, 2.7), (0, 3.1), (0, 21.4), (0, 37.34)	
FKwmBXII	12.8781, 1.2255, 1.6653, 1.416, 3.734	386.97, 397.40, 388.10, 391.10
	(3.44), (0.134), (0.0342), (0.09), (1.17)	
	(6,19.6), (0.8,1.5), (1.6,1.7), (1.3,1.58), (1.4,6.1)	
GOLLBXII	**5.297, 1.205, 8.634, 0.040, ---**	**386.803, 395.11, 387.54, 390.05**
	0.000, 0.000, 0.000, (0.029) ---	
	---, ---, ---, (0, 0.1), ---	

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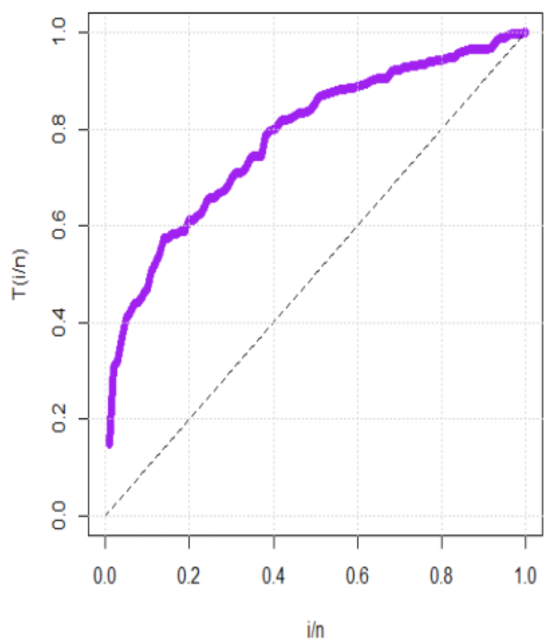
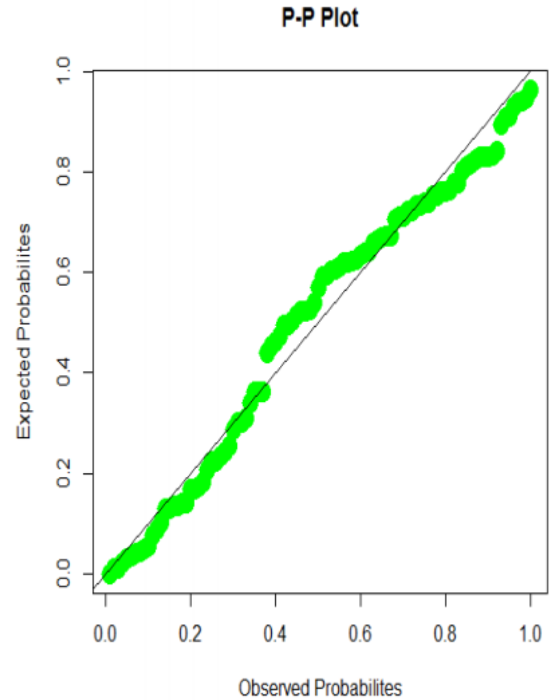
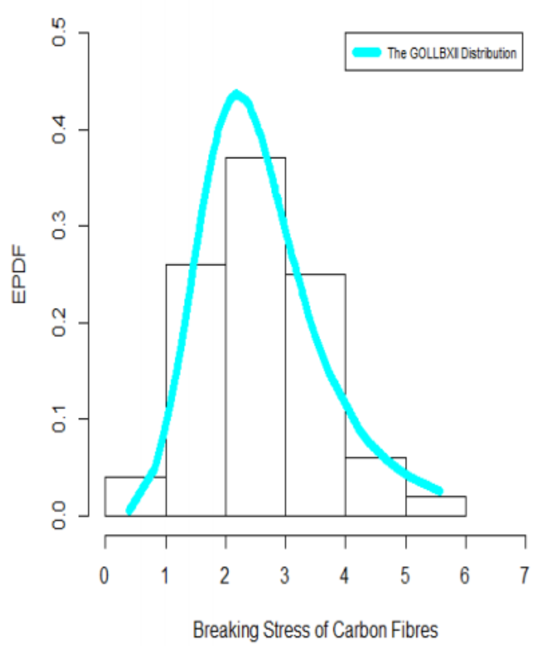


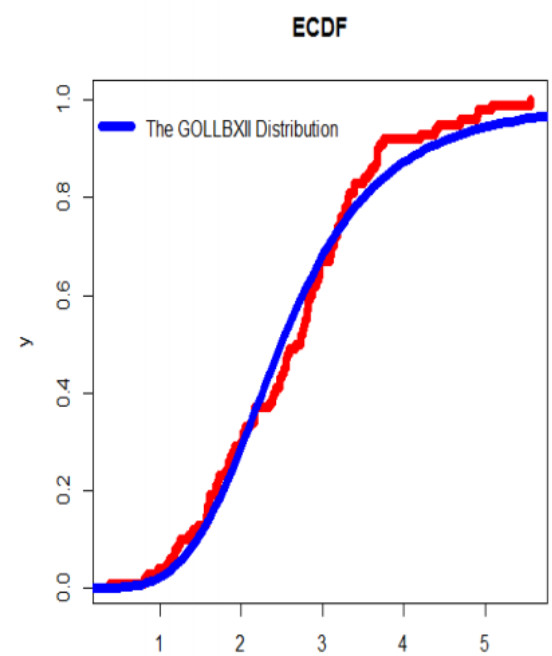
TTT plot



P-P plot

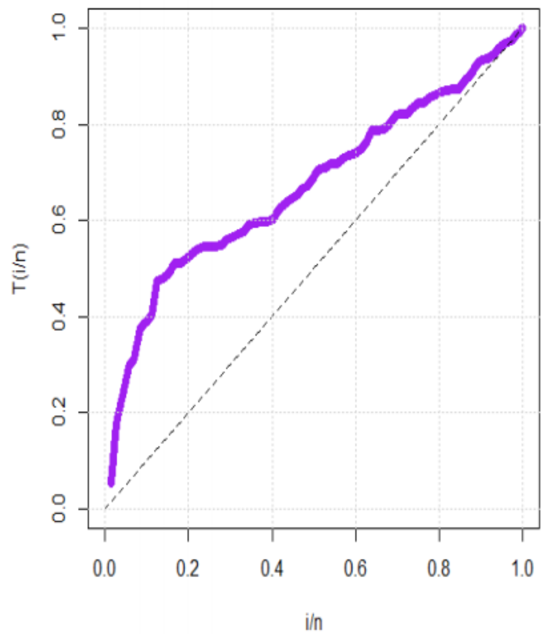


EPDF

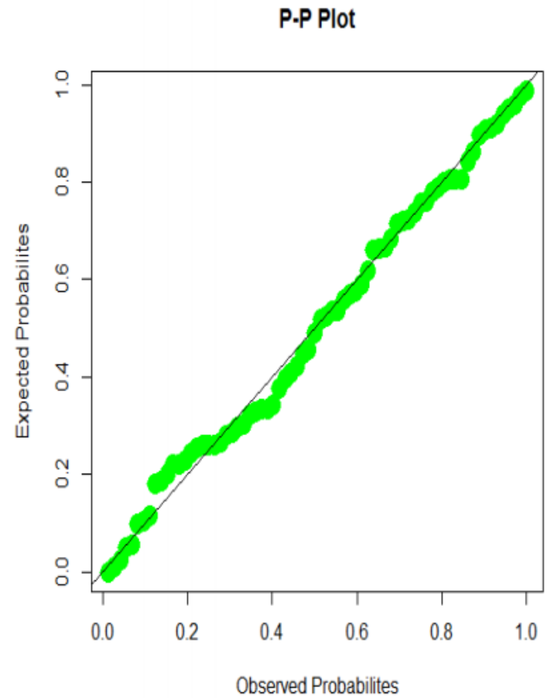


ECDF

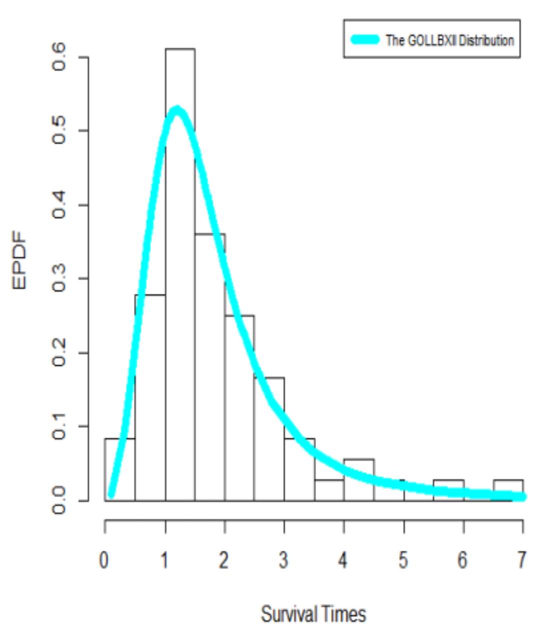
Figure 5: Plots for data set I.



TTT plot



P-P plot



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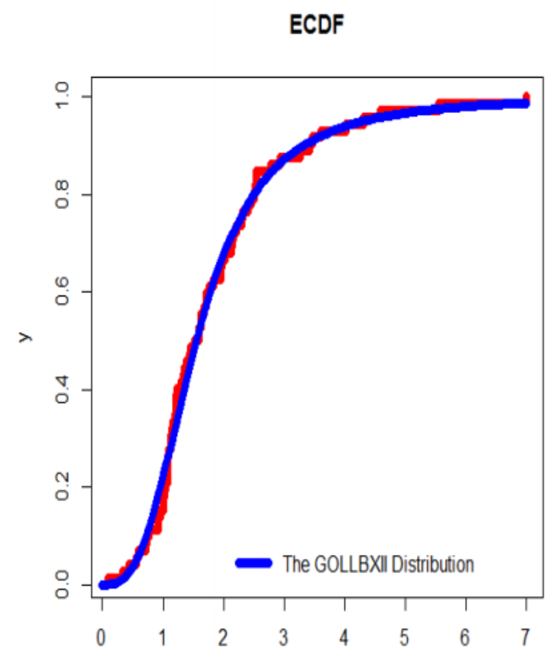

ECDF

Figure 6: Plots for data set II.

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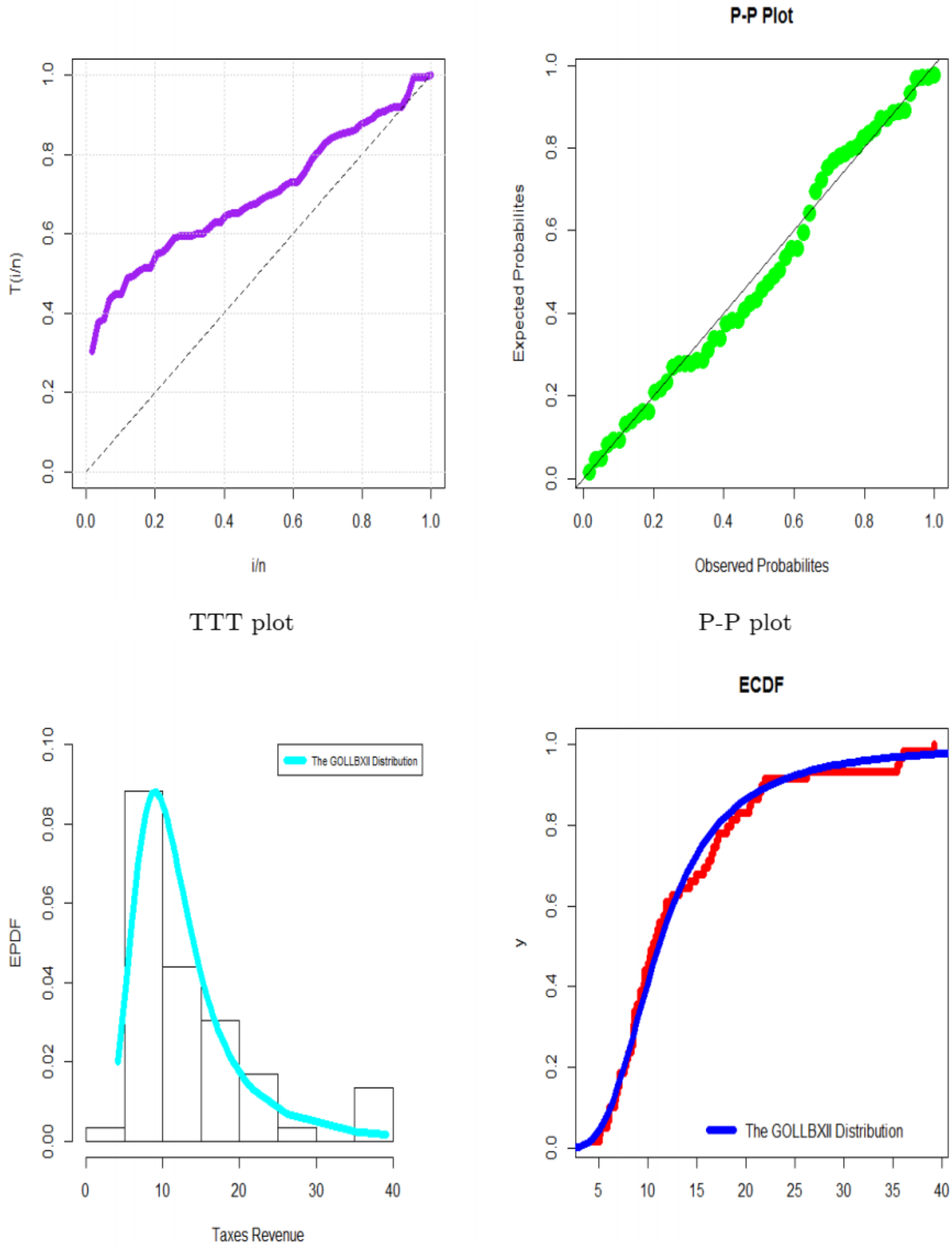

TTT plot P-P plot

EPDF ECDF

Figure 7: Plots for data set III.

6. Conclusions, discussion and future points

A new generalization of the Burr type XII model called the generalized odd log-logistic Burr type XII (GOLLBXII) model is introduced and studied. The genesis of the new Burr type XII model is derived based on the generalized odd log-logistic-G family. The new model generalizes at least eight important sub-models. Some useful properties related to the new model are derived. The new additional shape parameters α and β has an effect on mean, variance, skewness and kurtosis. For the GOLLBXII model, skewness can range in the interval ($-$ 134.7817, 26.46451). However, for the BXII model, skewness can range in the interval ($-$ 0.974193, 18.60987). For the GOLLBXII model, kurtosis can range in the interval ($-$ 181.8733, 3011661). However, for the BXII model, kurtosis can range in the interval (2.880286, 2139.368). The Meijer **G** function is used to derive the moment generating function. The copula-based construction is used to generate many bivariate and multivariate type distributions. We used the maximum likelihood method in estimation process. Graphically, we performed the simulation experiments to assess of the finite sample behavior of the MLEs, the biases for each parameter decrease to zero as $n \to \infty$, the MSEs for each parameter decrease to zero as $n \to \infty$. Three numerical examples are given for comparing models. The GOLLBXII has an enough flexibility for modeling different real data sets. Future works may be allocated to study the new bivariate and multivariate type distributions.

Introducing a new probability distribution is not difficult, but the real difficulty lies in presenting a new distribution that is flexible enough. The flexibility of probability distributions can be viewed from many aspects. One of these aspects, and the first of them, is the breadth of the density function and its inclusion of many forms that help in statistical modeling processes for different data. The second of these is the flexibility of the failure rate function of the new distribution. Also among these aspects is the ability of the new distribution to model real data. In order to assess whether a new probability distribution is suitable for the assignment at hand when examining its flexibility, it is crucial to take into account the characteristics of the data being modelled and to compare the new distribution to other widely used distributions. A distribution's adaptability can also be increased by mixing different distributions in a mixture model or by modeling non-typical data patterns with transformations. The new distribution has proven a high applied ability in statistical modeling operations through a range of applications on engineering and medical data. And the new distribution proved that it is the best among many other distributions closely related to it and competing with it. We do not claim that the new distribution is absolutely the best in the statistical literature. Because each statistical model has advantages and disadvantages, these advantages and disadvantages may relate to some issues, including the scope of application of the distribution and its flexibility in the estimation and modeling processes. It is intuitively known that there is no suitable statistical model for all types of data. Each statistical model has limitations and conditions when applied.

Below we will add some future point along with their corresponding references for helping readers to expand this work:

- **I.** Stress-strength reliability model under the GOLLBXII distribution (see Rasekhi et al. (2020) and Saber et al. (2022a,b,c,d)).
- **II.** Nikulin-Rao-Robson goodness-of-fit test under the GOLLBXII distribution (see Goual and Yousof (2020) and Goual et al. (2019, 2020), Yadav et al. (2020, 2022)).
- **III.** Censored and uncensored validation under the Bagdonavičius–Nikulin goodness-of-fit test (Ibrahim et al. (2019), Aidi et al. (2021), Ibrahim et al. (2021a, 2022), Khalil et al. (2023), Mansour et al. (2020a.b) and Yousof (2022a,b,c)).
- **IV.** Assessing and analyzing actuarial risks using insurance and reinsurance data via the new GOLLBXII distribution (see Mohamed et al. (2022a,b,c), Emam et al. (2023a), Hamed et al. (2022) and Yousof et al. (2023)).
- **V.** Provide a new discrete distribution based on the new distribution (see Aboraya et al. (2020), Ibrahim et al. (2021b), Eliwa et al. (2022), Chesneau et al. (2022) Yousof (2021c), Emam et al. (2023a)).

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